Aspect-ratio dependence of percolation probability in a rectangular system

Shinsuke Tsubakihara
Department of Applied Physics, Faculty of Science, Fukuoka University, Jonan-ku, Fukuoka 814-0180, Japan
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I investigate site percolation on a rectangular system (aspect ratio a) of a square lattice for a given occupation probability p (not restricted to p_c) using computer simulations. The dependence of the percolation probability R on a is shown and analyzed on the basis of a modified finite-size scaling function. A method for evaluating R without statistical simulations is proposed for given conditions (longitudinal dimension L, a, and p) of the system.

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The research on percolation in a rectangular system was initiated by Monetti and Albano [1–3]. Their work was further advanced by Cardy [4], Langlands and co-workers [5,6], Ziff [7–10], and Hovi and Aharony [11]. These researchers carried out computer simulations on two-dimensional (2D) lattices, and succeeded in deducing a function for evaluating the percolation probability R in a given aspect ratio and in the vicinity of the percolation threshold p_c. Since then, research on percolation in a three-dimensional rectangular system was performed recently [12–15]. In contrast, percolation on a rectangular system with a given occupation probability p (not restricted to p_c) has not yet been studied. Hence there is no method for evaluating R in such a system, though it is applicable to a wide range of actual percolation systems. In this report, I show the results of computer simulations performed on a 2D rectangular system with a given p, and analyze them based on a modified finite-size scaling function. Furthermore, I propose a method for evaluating R under given conditions (size, aspect ratio, and occupation probability) without carrying out statistical simulations.

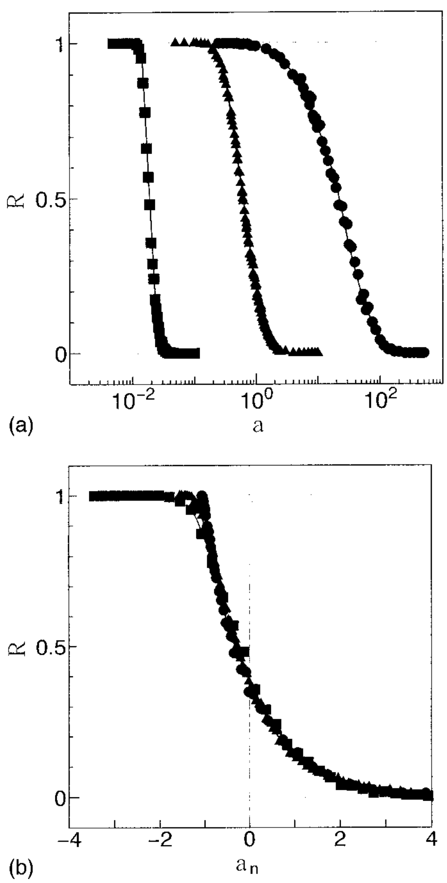

FIG. 1. (a) Variation of percolation probability R with aspect ratio a:(●)$L=20$ and $p=0.70$; (▲)$L=100$ and $p=0.58$; and (■)$L=1000$ and $p=0.45$. The number of trials for each value of a is 1000. (b) Variation of R with the transformed aspect ratio a_n. The symbols correspond to those in (a). Except for the portion where R begins to decrease from 1, the three curves are almost superimposable on top of one another. $a_n=0$ agrees with the average point of the transition region.

Computer simulations were carried out on square lattices (the site percolation model). The longitudinal and transverse dimensions of the rectangular system are denoted by L and M, respectively. The aspect ratio a is defined as

$$a=M/L. \tag{1}$$

Recently Acharyya and Stauffer reported that the value of R is different according to boundary conditions [16]. In this

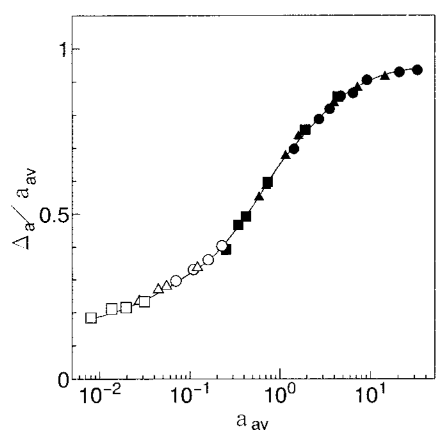

FIG. 2. Plot of $\Delta_a/a_{\rm av}$ vs $a_{\rm av}$:(●)$L=20$ and $p=0.60–0.70$;(▲)$L=50$ and $p=0.56–0.64$;(■)$L=100$ and $p=0.53–0.61$;(○)$L=200$ and $p=0.45–0.55$;(△)$L=500$ and $p=0.42–0.55$;(□) and $L=1000$ and $p=0.30–0.50$. These are based on R-a curves which become superimposable on the R-a_n curves of Fig. 1(b) through coordinate transformation.

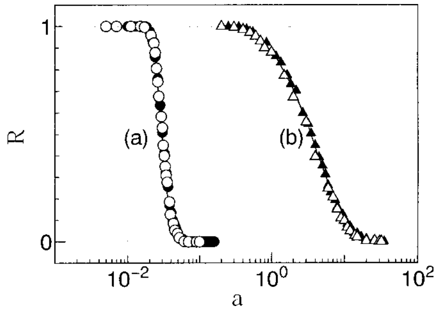

FIG. 3. R-a curves drawn for two values of $(p-p_c)L^{1/\nu}$. Two sets of L and p were chosen for each value. (a) $(p-p_c)L^{1/\nu}\sim -16.3$:(●)$L=500$ and $p=0.44$; (○)$L=1000$ and $p=0.50$. (b) $(p-p_c)L^{1/\nu}\sim 0.54$: (▲)$L=20$ and $p=0.65$; and (△)$L=100$ and $p=0.61$.

work, simulations were carried out under the free boundary condition, because it applies to many actual percolation systems. Each site of the square lattice was occupied randomly with a probability p. Percolation was defined when one or more clusters of occupied sites connected from the left side to the right side of the system. For a set of L, M, and p, 1000 different configurations were tried. R was calculated as the proportion of percolation frequency in the trials. The dependence of R on a was examined by changing M for a set of L and p.

The dependence of R on a is shown for three sets of L and p in Fig. 1(a). R decreases from 1 to 0 with a. For the R-a curve, I define the average point a_{av} and width Δ_a of the transition region as

$$a_{\mathrm{av}}=-\int_0^\infty a(dR/da)da, \tag{2}$$

$$\Delta_a^2=-\int_0^\infty (a-a_{\mathrm{av}})^2(dR/da)da. \tag{3}$$

Then I calculate a transformed aspect ratio a_n according to

$$a_n=(a/a_{\mathrm{av}}-1)/(\Delta_a/a_{\mathrm{av}}). \tag{4}$$

The coordinate transformation of a into a_n changes a_{av} and Δ_a of the R-a curve to 0 and 1, respectively. By the transformation, the R-a curves shown in Fig. 1(a) result in the curves shown in Fig. 1(b). The curves are superimposable on one another in spite of the great differences in L and p. I confirmed that all simulation curves transformed by Eq. (4) were superimposable on one another. Figure 2 illustrates a sigmoid curve representing the relationship between Δ_a/a_{av} and a_{av}, which are obtained from the simulation curves. The value of Δ_a/a_{av} increases with a_{av}. It must be noted that R decreases sharply with a in the transition region if the value is small. For example, an increase of a by about three times causes R to decrease from 1 to 0 if R is equal to 0.2.

The finite-size scaling theory is applied for the finite-size square system [17]. I modify the scaling function to the rectangular system as

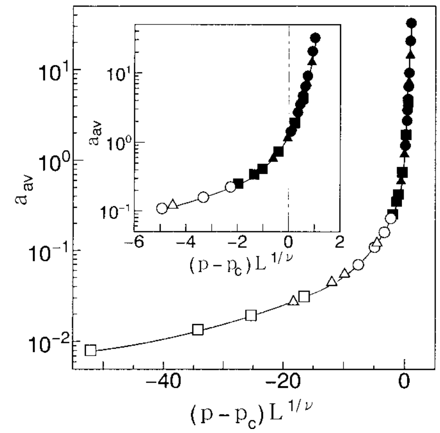

FIG. 4. Plot of a_{av} vs $(p-p_c)L^{1/\nu}$. The symbols here correspond to those of Fig. 2.

$$R=\Phi[a,(p-p_c)L^{1/\nu}], \tag{5}$$

where Φ is the generalized scaling function, and ν is the correlation length exponent. The values of p_c and ν are 0.592746 and $\frac{4}{3}$, respectively, in the case of site percolation on a square lattice [17]. According to Eq. (5), an identical R-a curve should be drawn for different sets of L and p if the $(p-p_c)L^{1/\nu}$ values of the sets are identical. R-a curves drawn for two values of $(p-p_c)L^{1/\nu}$ are shown in Fig. 3. The two curves in the case of (a) $(p-p_c)L^{1/\nu}\sim -16.3$ are superimposable on top of each other even though they are drawn for different sets of L and p. Similarly, the two curves in the case of (b) $(p-p_c)L^{1/\nu}\sim 0.54$ are superimposable on top of each other. These results attest to the correctness of Eq. (5); accordingly, I analyze the results of simulations based on it. It is an important point that Eq. (5) can be widely applied as shown in this figure; in other words, it is not restricted to the condition that L and p approach infinity and p_c, respectively. Further, Eq. (5) reveals that a_{av} is a function of $(p-p_c)L^{1/\nu}$ on the grounds that the R-a curve is determined by the value of $(p-p_c)L^{1/\nu}$. Figure 4 shows the relationship between a_{av} and $(p-p_c)L^{1/\nu}$. The value of a_{av} increases monotonically with $(p-p_c)L^{1/\nu}$, and the increasing rate becomes abruptly large in the neighborhood of 0.

On the basis of the above results, it is possible to evaluate R in the following way when L, a, and p of the system are known. The values of a_{av} and Δ_a are first obtained from the curves of Figs. 2 and 4. These values are those of the R-a curve drawn for the set of L and p. Then the transformed aspect ratio a_n is calculated from a by substituting the values of a_{av} and Δ_a into Eq. (4). Finally, R is evaluated from a_n by referring to the curve of Fig. 1(b). The validity of this method was confirmed by applying it to arbitrarily chosen conditions ($L=150$, $a=0.26$, and $p=0.541$); that is, $R=0.286$ (evaluated value) agreed well with $R=0.283$ (simulated value). This method makes it possible for us to evaluate R of the rectangular system without carrying out statistical simulations.

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